1. Solving Linear Equations

1.1 Solutions of Linear Equations

A common problem I run into is students not fully understanding the difference between an algebraic expression and an equation. Simply put, an equation will contain an equal sign, while an expression does not contain an equal sign. To solve an equation, we are trying to determine what numbers we can substitute into the variable to make a true statement.

As an example, consider the equation

$$5x - 3 = 2x + 6$$

By substituting in 3 for x, we would get

$$\begin{aligned} 5(3) - 3 &= 2(3) + 6 \\ 15 - 3 &= 6 + 6 \\ 12 &= 12 \end{aligned}$$

Since this is a true statement, we can say $x = 3$ is a solution. On the other hand, substituting in -2 for x gives

$$\begin{aligned} 5(-2) - 3 &= 2(-2) + 6 \\ -10 - 3 &= -4 + 6 \\ -13 &= 2 \end{aligned}$$

Since this is a false statement, we can say that $x = -2$ is not a solution to this equation.

■ **Example 1.1** Determine if $x = -2$ or $x = 4$ are solutions to the equation

$$x^2 - 8 = 3x + 2$$

Solution Substituting $x = -2$ into this equation gives

$$\begin{aligned} (-2)^2 - 8 &= 3(-2) + 2 \\ 4 - 8 &= -6 + 2 \\ -4 &= -4 \end{aligned}$$

Since this is a true statement, we can say that $x = -2$ is a solution of the given equation.

Substituting $x = 4$ into this equation gives

$$\begin{aligned} 4^2 - 8 &= 3(4) + 2 \\ 16 - 8 &= 12 + 2 \\ 8 &= 14 \end{aligned}$$

Since this is not a true statement, we can say that $x = 4$ is not a solution of the given equation. ■

■ **Example 1.2** Determine if $x = -1$ or $x = 5$ are solutions to the equation

$$x^2 - 8 = 3x + 2$$

Solution Substituting $x = -1$ into this equation gives

$$\begin{aligned} (-1)^2 - 8 &= 3(-1) + 2 \\ 1 - 8 &= -3 + 2 \\ -7 &= -1 \end{aligned}$$

Since this is not a true statement, we can say that $x = -1$ is not a solution of the given equation. Substituting $x = 5$ into this equation gives

$$\begin{aligned} 5^2 - 8 &= 3(5) + 2 \\ 25 - 8 &= 15 + 2 \\ 17 &= 17 \end{aligned}$$

Since this is a true statement, we can say that $x = 5$ is a solution of the given equation. ■

1.2 Solving Linear Equations

We start solving equations with linear equations. These are anything that can be written in the form

$$ax + b = 0$$

where a and b are real numbers with $a \neq 0$, and x is the variable. Essentially, we should not raise the variable to any power other than 1.

■ **Example 1.3** Each of the following equations are linear.

- $4x = \frac{2}{3}x + 1$
- $x + 7 = \frac{x}{4}$
- $3x + 7 = -2$

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■ **Example 1.4** Each of the following equations are not linear.

- $x^2 + 7x = 2$
- $\sqrt{x} + 5x = 12$
- $\frac{2}{x} - 3x = 4$

The first of these equations is not linear since x is raised to the second power (that is, x^2). The second of these is not linear since x is raised to an exponent of $1/2$ (that is $\sqrt{x} = x^{1/2}$). The final equation is not linear since x is raised to an exponent of -1 (that is, $2/x = 2x^{-1}$). ■

When solving this type of equation, there are two real rules of equality:

- You can add/subtract the same number from both sides.
- You can multiply/divide by the same non-zero number on both sides.

In general, when solving linear equations, we start by simplifying, if possible. This can include clearing parentheses, combining like terms on the same side of the equation, or multiplying through by the LCD to clear fractions. After this, we try to collect all terms with the variable on one side, and all constant terms on the other. We then divide by the coefficient of the variable to isolate the variable.

Definition 1.2.1 — Algorithm for Solving Linear Equations.

1. Simplify, if possible. This may include:
   - Clearing parentheses
   - Combining like terms on the same side of an equation
   - Multiplying through by the LCD to clear fractions.
2. Collect all terms with the variable on one side of the equation.
3. Collect all constant terms on the other side of the equation.
4. Divide by the coefficient of the variable to isolate the variable.

Example 1.5 Solve the given equation

$$2x-3=-25$$

Solution We can see that this equation only has a single term containing the variable. Thus, we start by getting the constants on the other side of the equation.

$$\begin{aligned} 2x-3 &= -25 \\ +3 \quad & \quad +3 \\ 2x &= -22 \end{aligned}$$

We then divide by the coefficient of the variable to get the final answer.

$$\begin{aligned} \frac{2x}{2} &= \frac{-22}{2} \\ x &= -11 \end{aligned}$$

This then gives the solution. We can substitute $x=-11$ into the original equation to check that our answer is correct.

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Example 1.6 Solve the given equation

$$3x+5=x-27$$

Solution The first step for this equation is to get the variables on one side of the equation.

$$\begin{aligned} 3x+5 &= \quad x-27 \\ -x \quad & \quad -x \\ 2x+5 &= -27 \end{aligned}$$

We then need to get all the constants on the other side of the equation.

$$\begin{aligned} 2x+5 &= -27 \\ -5 \quad & \quad -5 \\ 2x &= -32 \end{aligned}$$

Finally, dividing by the coefficient gives us the solution.

$$\begin{aligned} \frac{2x}{2} &= \frac{-32}{2} \\ x &= -16 \end{aligned}$$

This then gives the solution. We can substitute $x = -16$ into the original equation to check that our answer is correct.

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■ **Example 1.7** Solve the given equation

$$-(x+10) = 2(x+5)+31$$

Solution We need to start this equation by simplifying each side.

$$\begin{aligned} -(x+10) &= 2(x+5)+31 \\ -x-10 &= 2x+10+31 \\ -x-10 &= 2x+41 \end{aligned}$$

Next, we need to get all of the variables on one side of the equation.

$$\begin{aligned} -x-10 &= 2x+41 \\ +x \qquad &\quad\ +x \\ -10 &= 3x+41 \end{aligned}$$

We can then get the constants on the other side of the equation.

$$\begin{aligned} -10 &= 3x+41 \\ -41 \quad &\quad -41 \\ -51 &= 3x \end{aligned}$$

We now divide both sides by the coefficient to get the final solution.

$$\begin{aligned} \frac{-51}{3} &= \frac{3x}{3} \\ -17 &= x \end{aligned}$$

This then gives the solution. We can substitute $x = -17$ into the original equation to check that our answer is correct.

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■ **Example 1.8** Solve the given equation

$$\frac{x}{6} + \frac{2}{3} = \frac{3x}{4}$$

Solution We see that the LCD of these fractions is 12. Thus, we start by multiplying both sides of the equation by 12.

$$\begin{aligned} \left(\frac{x}{6} + \frac{2}{3}\right) \cdot 12 &= \left(\frac{3x}{4}\right) \cdot 12 \\ \frac{x}{6} \cdot 12 + \frac{2}{3} \cdot 12 &= \frac{3x}{4} \cdot 12 \\ 2x + 8 &= 9x \end{aligned}$$

Now this looks more like previous examples. We start by getting the variables on the same side of the equation.

$$\begin{aligned} 2x + 8 &= 9x \\ -2x \quad & \quad -2x \\ 8 &= 7x \end{aligned}$$

We divide by the coefficient to isolate the variable and get the final solution.

$$\begin{aligned} \frac{8}{7} &= \frac{7x}{7} \\ \frac{8}{7} &= x \end{aligned}$$

This then gives the solution. We can substitute $x = 8/7$ into the original equation to check that our answer is correct.

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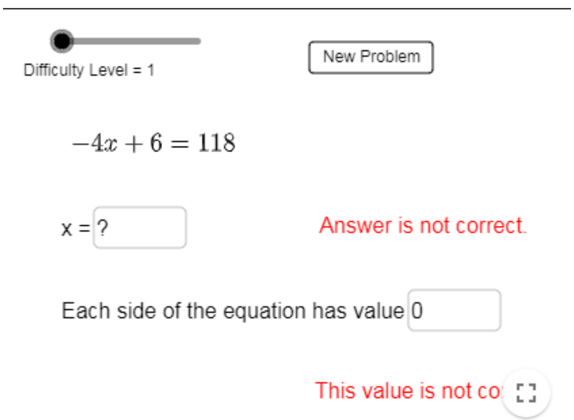

Click on the image to the left to access a Geogebra applet for further practice solving linear equations. Alternatively, this applet is available at the website https://ggbm.at/xusjrzb9.