3. Graphing Lines

3.1 Slope of a Line

When discussing lines, the first thing to consider is the slope of the line. This is essentially a measure of how steep the line is. We call the distance we move horizontally the run. The corresponding vertical change is known as the rise. Using these terms, we have that

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

Whenever a line moves us as you move to the right, we consider the rise to be positive. If the line moves down as you move to the right, we consider the rise to be negative. This allows us to distinguish between both the steepness of the line and whether it is "uphill" or "downhill".

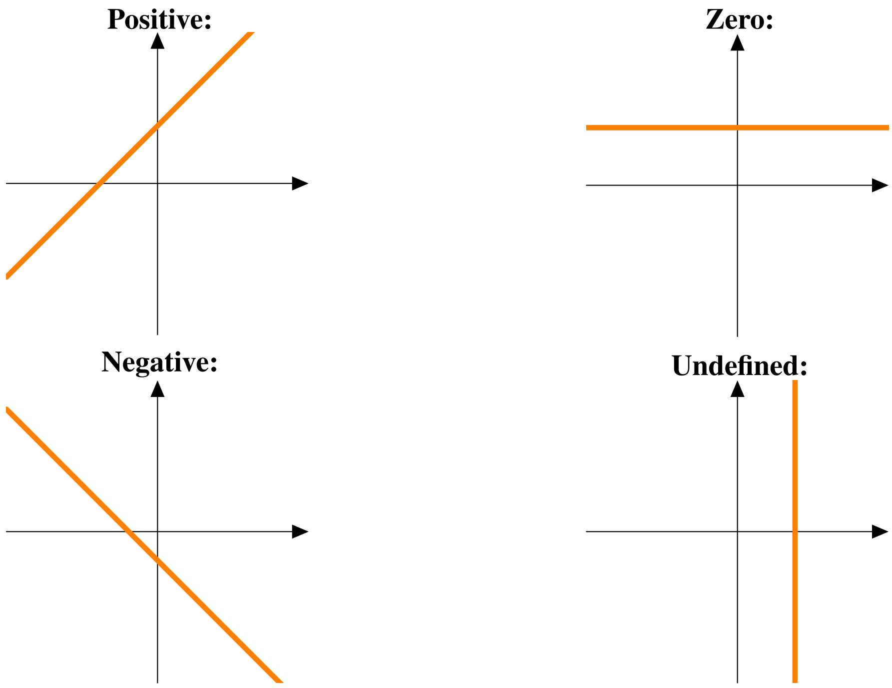
Let's suppose we have two points: $A = (x_1, y_1)$ and $B = (x_2, y_2)$. We want to try to find the slope of the line between them. We know that the run is the distance we move horizontally. Since the x-axis is the horizontal axis, we have that the run is given by $x_2 - x_1$. The corresponding vertical change, the rise, would then be related to the y-coordinates. This means that the rise is given by $y_2 - y_1$. This gives us another definition of slope.

Definition 3.1.1 The **slope**, denoted by m, of the line passing through points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

provided $x_1 \neq x_2$. In the case where $x_1 = x_2$, we say the slope is undefined.

This means there are four different possible slope types: positive, negative, zero, and undefined. We can see an example of each of those below:



Let's look at examples of calculating these values.

■ **Example 3.1** For each of the following, calculate the slope of the line passing through the given points.

1. $(-3,2)$, $(4,-1)$
2. $(1,-5)$, $(3,0)$
3. $(2,5)$, $(7,5)$
4. $(3,1)$, $(3,-2)$

Solution

1. We have that $x_1 = -3$, $y_1 = 2$, $x_2 = 4$ and $y_2 = -1$. Substituting these into the formula for slope gives
$$m = \frac{-1-2}{4-(-3)} = \frac{-1-2}{4+3} = \frac{-3}{7}$$
Thus, this is an example of something with negative slope.
2. We have that $x_1 = 1$, $y_1 = -5$, $x_2 = 3$, and $y_2 = 0$. Substituting these into the formula for slope gives
$$m = \frac{0-(-5)}{3-1} = \frac{5}{2}$$
Thus, this is an example of something with positive slope.
3. We have that $x_1 = 2$, $y_1 = 5$, $x_2 = 7$ and $y_2 = 5$. Substituting these into the formula for slope gives
$$m = \frac{5-5}{7-2} = \frac{0}{5} = 0$$
Thus, this is an example of something with zero slope. Note that a zero slope will happen exactly when $y_1 = y_2$.
4. For this one, we may notice that $x_1 = x_2$. According to the formula for slope, this means the slope is undefined. Let's use the formula to see what happens:
$$m = \frac{-2-1}{3-3} = \frac{-3}{0}$$
We know division by zero is undefined. This is further proof that the slope is undefined.

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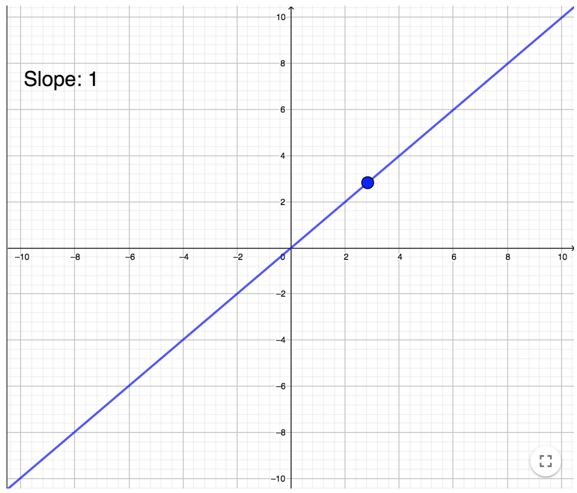


Click on the image to the left to access a Geogebra applet to help visualize how the line changes as the slope changes. Alternatively, this applet is available at the website https://ggbm.at/rgza4d8b.

3.2 Equations of Lines

Now that we have a better understanding of what slope is and how to calculate it, let's look at the full equation for a line. There are several forms of equations for lines. We start with the point-slope form.

Definition 3.2.1 Given the slope of a line, m, and a point on the line, (x_1, y_1), the **point-slope form** of the line can be found by

$$y - y_1 = m(x - x_1)$$

Notice that the name of this form of a line tells you exactly what you will need to find it: a point and the slope. Note that we can also use this form if we have two points on the line. This is because these two points will allow us to calculate slope, which then gives us everything we need.

■ **Example 3.2** Find the point-slope equation of a line with slope $2/3$ that passes through $(-2, -3)$. Then sketch this line.

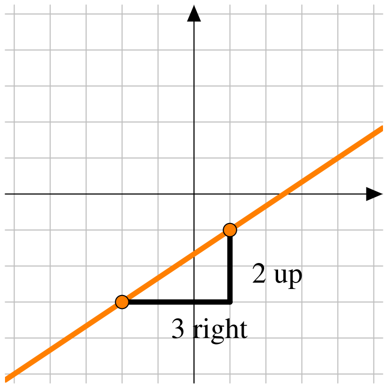
Solution We know that $m = \frac{2}{3}$, $x_1 = -2$ and $y_1 = -3$. Using the formula gives

$$y - (-3) = \frac{2}{3}(x - (-2))$$

Simplifying double negatives then gives

$$y + 3 = \frac{2}{3}(x + 2)$$

In order to graph this line, we will use the fact that we know one point on the graph, $(-2, -3)$, as well as the idea of the slope. The slope tells us that if we move 3 units in the x-direction, we should move 2 units in the y-direction.



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■ **Example 3.3** Find the point-slope equation of a line passing through $(1,-2)$ and $(3,2)$.
Solution Since we are given two points, we start by finding the slope:

$$m = \frac{2-(-2)}{3-1} = \frac{2+2}{3-1} = \frac{4}{2} = 2$$

We then pick a point to use for the point-slope form. We will use $(1,-2)$. This then gives

$$y-(-2) = 2(x-1)$$

or, equivalently,

$$y+2 = 2(x-1)$$

Note that if we would have used the point $(3,2)$ instead, we would have

$$y-2 = 2(x-3)$$

While these two solutions look very different, they are both correct and define the same line.

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Next, we look at the slope-intercept form of a line. As the name suggests, this equation uses the slope and the y-intercept of the graph.

Definition 3.2.2 Given the slope of a line m, and the y-intercept, $(0,b)$, the **slope-intercept form** of the line is given by

$$y = mx+b$$

■ **Example 3.4** Find an equation of the line with slope $-1/2$ and y-intercept $(0,4)$.
Solution We know that $m = -1/2$ and $b = 4$. Thus, the equation of the line is given by

$$y = -\frac{1}{2}x+4$$

■

We can also move any equation of a line to slope-intercept form to easily find the slope and y-intercept. Let's use this to show that the equations $y+2 = 2(x-1)$ and $y-2 = 2(x-3)$ that we found previously are actually the same.

■ **Example 3.5** Show that $y+2 = 2(x-1)$ and $y-2 = 2(x-3)$ are actually the same line.

Solution We start with $y+2=2(x-1)$. Using the distributive property, we have

$$y+2=2x-2$$

Subtracting 2 from both sides then gives the equation in slope-intercept form:

$$y=2x-4$$

Looking at the other equation, we once again start by using the distributive property:

$$y-2=2x-6$$

Adding 2 to both sides then gives:

$$y=2x-4$$

Thus, these are the same equation.

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While these formulas are very useful and can tell us a lot, there are two types of lines that they don't account for. The first is a horizontal line. We know that a horizontal line has a slope of 0. So in this case, the slope-intercept form becomes

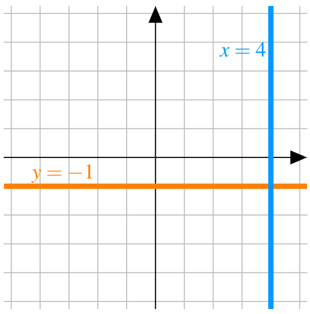
$$y=0x+b \qquad \rightarrow \qquad y=b$$

Thus, a horizontal line is always given by $y=b$, where b is the y-intercept of the equation.

On the other hand, a vertical line has undefined slope. In this case, we know that all of the x-coordinates are the same. Let's say the x-coordinate is always a. This means the equation of the vertical line is given by $x=a$.

■ **Example 3.6** Graph the equations $x=4$ and $y=-1$.

Solution We know $x=4$ is a vertical line passing through $(4,0)$ and $y=-1$ is a horizontal line passing through $(0,-1)$. We can see the graphs below.



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3.3 Perpendicular and Parallel Lines

Let's now consider parallel lines. As a reminder, lines are parallel if they never intersect. This means they will have the same slope.

Definition 3.3.1 Two lines are parallel if they have the same slope. In addition, if two lines have the same slope, then they are parallel.

■ **Example 3.7** Find the equation of the line that is parallel to $x-2y+8=0$ and passes through the point $(2,-4)$. Give the equation in slope-intercept form.

Solution We first need to find the slope of this line. Solving for y puts this line in slope-intercept form, which then allows us to find the slope. We start by adding $2y$ to both sides of the equation:

$$x+8=2y$$

We then divide both sides by 2:

$$\frac{1}{2}x+4=y$$

Thus, we have that the slope of this line is $\frac{1}{2}$. Since parallel lines have the same slope, the new line will also have slope $\frac{1}{2}$. Using point-slope form, we have

$$y-(-4)=\frac{1}{2}(x-2)$$

Using the distributive property, we have

$$y+4=\frac{1}{2}x-1$$

Subtracting 4 from both sides then gives

$$y=\frac{1}{2}x-5$$

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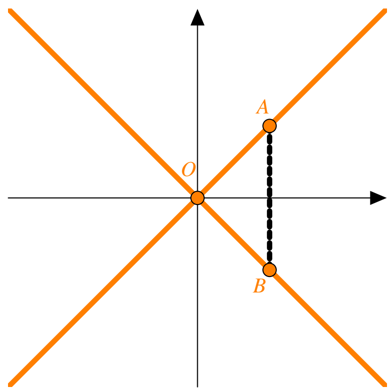
Next, we consider perpendicular lines. Let's say the slopes of two perpendicular lines are m_1 and m_2. Find lines parallel to these that pass through the origin. This makes the intersection point of the perpendicular lines occur at the origin. Consider the points $A=(1,m_1)$ on the first line and $B=(1,m_2)$ on the second line. Next, we draw the line segment between A and B, as seen in the diagram below.



We have now formed a right triangle, with hypothenuse given by the line segment between A and B. The Pythagorean theorem then gives

$$d(O,A)^2+d(O,B)^2=d(A,B)^2$$

Using the distance formula gives

$$\begin{aligned}
\left(\sqrt{(1-0)^2+(m_1-0)^2}\right)^2+\left(\sqrt{(1-0)^2+(m_2-0)^2}\right)^2 &= \left(\sqrt{(1-1)^2+(m_2-m_1)^2}\right)^2 \\
(1^2+m_1^2)+(1^2+m_2^2) &= (1-1)^2+(m_2-m_1)^2 \\
2+m_1^2+m_2^2 &= m_2^2-2m_1m_2+m_1^2 \\
2 &= -2m_1m_2 \\
-1 &= m_1m_2
\end{aligned}$$

Thus, we have discovered that the slopes of two perpendicular lines must multiply to be -1. This is often given in the alternative form, $m_2 = -\dfrac{1}{m_1}$. This basically says that the slopes of perpendicular lines will be negative reciprocals of one another.

Definition 3.3.2 Two lines with slopes m_1 and m_2 are perpendicular if $m_1 m_2 = -1$, that is, their slopes are negative reciprocals. In addition, if the slopes of two lines are negative reciprocals, then they are perpendicular. Also, horizontal lines are perpendicular to vertical lines.

■ **Example 3.8** Find the equation of the line that is perpendicular to $x-2y+8=0$ and passes through the point $(2,-4)$. Give the equation in slope-intercept form.

Solution We first need to find the slope of this line. Solving for y puts this line in slope-intercept form, which then allows us to find the slope. We start by adding $2y$ to both sides of the equation:

$$x+8=2y$$

We then divide both sides by 2:

$$\frac{1}{2}x+4=y$$

Thus, we have that the slope of this line is $\dfrac{1}{2}$. Since perpendicular slopes are negative reciprocals, the new line will have slope of -2. Using point-slope form, we have

$$y-(-4)=-2(x-2)$$

Using the distributive property, we have

$$y+4=-2x+4$$

Subtracting 4 from both sides then gives

$$y=-2x$$

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